SOLUTION OF THE INVERSE BOUNDARY-VALUE PROBLEM OF HEAT CONDUCTION IN AN OVERDEFINED FORMULATION

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An interaction scheme is considered for the solution of a nonlinear inverse heat-conduction problem with the results of measuring the temperature at an arbitrary number of points within the body taken into account.

In solving inverse heat-conduction problems (IHCP), formulations of the IHCP with the least number of temperature sensors needed from which the information will assure uniqueness of the solution of the problem are distinguished from overdefined formulations when the number of temperature sensors installed is greater than is required from the uniqueness condition. Installation of additional sensors permits more complete information to be obtained about the thermal state of the body and the error in determining the temperatures and thermal fluxes on its surface from the solution of the IHCP to be reduced. It is here expedient to use an extremal form of the problem with an iterative principle for regularization of the solution [1].

Let us consider a boundary-value inverse heat-conduction problem for an infinite plate with moving boundaries within which n temperature sensors are placed at different distances $x = X_i(\tau)$, $i = \overline{1, n}$ from the boundary $X_0(\tau)$, where the results of their measurements $f_i(\tau)$, $i = \overline{1, n}$ are the experimental dependences:

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\ X_0(\tau) < x < X_n(\tau),\ \tau > 0, \quad (1)$$

$$T(x,\ 0) = \varphi(x),\ X_0(0) \leqslant x \leqslant X_n(0), \quad (2)$$

$$-\lambda(T)\frac{\partial T(X_0(\tau),\ \tau)}{\partial x} = q_0(\tau), \quad (3)$$

$$-\lambda(T)\frac{\partial T(X_n(\tau),\ \tau)}{\partial x} = q_n(\tau), \quad (4)$$

$$T(X_i(\tau),\ \tau) = f_i(\tau),\ i = \overline{1, n},\ X_0(\tau) < X_1(\tau) < \ldots < X_n(\tau), \quad (5)$$

where $C(T)$, $\lambda(T)$, $\varphi(x)$, $q_n(\tau)$, $f_i(\tau)$, $i = \overline{1, n}$, $X_i(\tau)$, $i = \overline{0, n}$ are known functions.

Determine the heat flux density $q_0(\tau)$ and the temperature field in the plate.

For convenience in the numerical realization, the problem is solved in a coordinate system coupled to the moving boundaries $X_i(\tau) - t = \tau$, $z_i = [x - X_{i-1}(\tau)]/[X_i(\tau) - X_{i-1}(\tau)]$, $i = \overline{1, n}$, hence (1)-(5) become

$$C(T)\frac{\partial T^i}{\partial t} = \frac{1}{l_i^2(t)}\frac{\partial}{\partial z_i}\left[\lambda(T)\frac{\partial T^i}{\partial z_i}\right] + \frac{C(T)}{l_i(t)}[\dot{X}_{i-1}(t) + z_i\dot{l}_i(t)]\frac{\partial T^i}{\partial z_i},\ 0 < z_i < 1,\ i = \overline{1, n},\ t > 0, \quad (6)$$

$$T^i(z_i,\ 0) = \varphi_i(z_i),\ 0 \leqslant z_i \leqslant 1,\ i = \overline{1, n}, \quad (7)$$

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$$-\frac{\lambda(T)}{l_1(t)}\frac{\partial T^1(0,\ t)}{\partial z_1}=q_0(t), \tag{8}$$

$$-\frac{\lambda(T)}{l_n(t)}\frac{\partial T^n(1,\ t)}{\partial z_n}=q_n(t), \tag{9}$$

$$T^i(1,\ t)=T^{i+1}(0,\ t),\ i=\overline{1,\ n-1}, \tag{10}$$

$$\frac{1}{l_i(t)}\frac{\partial T^i(1,\ t)}{\partial z_i}=\frac{1}{l_{i+1}(t)}\frac{\partial T^{i+1}(0,\ t)}{\partial z_{i+1}},\ i=\overline{1,\ n-1}, \tag{11}$$

$$T^i(1,\ t)=f_i(t),\ i=\overline{1,\ n}, \tag{12}$$

where $l_i(t)=X_i(t)-X_{i-1}(t),\ \dot{l}_i(t)=dl_i(t)/dt.$

The desired function $q_0(t)$ is determined from the requirement of a minimum of the rms residual

$$J(q_0)=\frac{1}{2}\sum_{i=1}^{n}\int_0^{t_p}[T^i(1,\ t)-f_i(t)]^2dt. \tag{13}$$

The search for the solution of IHCP is performed by the scheme of the method of conjugate gradients in which the gradient of the functional (13) is evaluated by a formula based on the solution of the boundary-value problem adjoint to the problem (6)-(11).

Considering $q_0(t)$ as a certain control function minimizing functional (13), by following the methodology elucidated in [2], the equation of this problem can be written as

$$-\frac{\partial\Psi^i}{\partial t}=\frac{\partial^2}{\partial z_i^2}(A^i\Psi^i)-\frac{\partial}{\partial z_i}(B^i\Psi^i)+D^i\Psi^i, \tag{14}$$

$$\begin{gathered}0<z_i<1,\ 0\leqslant t<t_p,\ i=\overline{1,\ n},\\ \Psi^i(z_i,\ t_p)=0,\ 0\leqslant z_i\leqslant 1,\ i=\overline{1,\ n},\end{gathered} \tag{15}$$

$$\left[\frac{\partial\lambda^1(0,\ t)}{\partial z_1}\frac{A^1(0,\ t)}{\lambda^1(0,\ t)}-B^1(0,t)\right]\Psi^1(0,\ t)+\frac{\partial}{\partial z_1}[A^1(0,\ t)\Psi^1(0,\ t)]=0, \tag{16}$$

$$\left[\frac{\partial\lambda^n(1,\ t)}{\partial z_n}\frac{A^n(1,\ t)}{\lambda^n(1,\ t)}-B^n(1,\ t)\right]\Psi^n(1,\ t)+\frac{\partial}{\partial z_n}[A^n(1,\ t)\Psi^n(1,t)]=[T^n(1,\ t)-f_n(t)], \tag{17}$$

$$\frac{\Psi^i(1,\ t)}{l_i(t)}=\frac{\Psi^{i+1}(0,\ t)}{l_{i+1}(t)},\ i=\overline{1,\ n-1}, \tag{18}$$

$$\begin{gathered}\left[\frac{\partial\lambda^i(1,\ t)}{\partial z_i}\frac{A^i(1,\ t)}{\lambda^i(1,\ t)}-B^i(1,t)\right]\Psi^i(1,\ t)+\frac{\partial}{\partial z_i}[A^i(1,\ t)\Psi^i(1,\ t)]-\\ -\left[\frac{\partial\lambda^{i+1}(0,\ t)}{\partial z_{i+1}}\frac{A^{i+1}(0,\ t)}{\lambda^{i+1}(0,\ t)}-B^{i+1}(0,t)\right]\Psi^{i+1}(0,\ t)-\frac{\partial}{\partial z_{i+1}}[A^{i+1}(0,\ t)\Psi^{i+1}(0,\ t)]=[T^i(1,\ t)-f_i(t)],\\ i=\overline{1,\ n-1},\end{gathered} \tag{19}$$

where

$$A^i(z_i,\ t)=\lambda^i(z_i,\ t)/C^i(z_i,\ t)/l_i^2(t),\ B^i(z_i,t)=\left[2\frac{\partial\lambda^i(z_i,\ t)}{\partial z_i}+C^i(z_i,\ t)(\dot{X}_{i-1}+z_i\dot{l}_i)\,l_i(t)\right]/C^i(z_i,\ t)/l_i^2(t),$$

$$D^i(z_i, t) = \left[\frac{\partial^2\lambda^i(z_i, t)}{\partial z_i^2} + \frac{\partial C^i(z_i, t)}{\partial z_i}(\dot{X}_{i-1} + z_i\dot{l}_i)\, l_i(t) - \frac{\partial C^i(z_i, t)}{\partial t}\, l_i^2\right] \Big/ C^i(z_i, t)/l_i^2(t),$$

and the formula to compute the gradient also is

$$J'_{q_0} = \frac{A^1(0, t)}{\lambda^1(0, t)}\, \Psi^1(0, t)\, l_1(t). \tag{20}$$

In computing the gradient by means of (20), the error in the solution of the IHCP will be determined to a significant degree by the selection of the initial approximation. In this connection, a formula is used in the search scheme to compute the gradient relative to the first derivative of the thermal flux density with respect to the time [1-3]:

$$J'_{\dot{q}_0} = \int_t^{t_p} \frac{A^1(0, \tau)}{\lambda^1(0, \tau)}\, \Psi^1(0, \tau)\, l_1(\tau)\, d\tau, \tag{21}$$

where $\dot{q}_0 = dq_0/dt$.

The iteration scheme of the conjugate gradients method is here written in the form

$$q_0^{s+1}(t) = q_0^s(t) - \alpha r^s(t), \quad s = 0, 1, 2, \ldots, \quad r^s(t) = q^0(0) + \int_0^t p^s(\tau)\, d\tau,$$
$$p^s(t) = -J'^{\,s}_{\dot{q}_0} + \beta^s p^{s-1}(t), \quad \beta^s = (J'^{\,s}_{\dot{q}_0} - J'^{\,s-1}_{\dot{q}_0}, J'^{\,s}_{\dot{q}_0})/(J'^{\,s-1}_{\dot{q}_0}, J'^{\,s-1}_{\dot{q}_0}), \quad \beta^0 = 0, \tag{22}$$

where $q^0(t)$ is a known initial approximation.

The thermal flux density at the initial time $q^0(0)$ is assumed known.

The magnitude of the step α in going from the s-th to the (s + 1)-th approximation is found from the condition $\min_\alpha J(q_0^s - \alpha r^s)$. Using a linear estimate of functional (13) in the (s + 1)-th iteration

$$J(q_0^{s+1}) = \frac{1}{2}\sum_{i=1}^{n}\int_0^{t_p}[T^i(1, t, q_0^s - \alpha r^s) - f_i(\tau)]^2 dt = \frac{1}{2}\sum_{i=1}^{n}\int_0^{t_p}[T^i(1, t, q_0^s) - \alpha\vartheta^i(1, t, r^s) - f_i(t)]^2 dt, \tag{23}$$

where $\vartheta^i(1, t, r^s)$, $i = \overline{1, n}$ are the temperature variations due to variations in the thermal flux density r^s, an effective procedure for determining α can be constructed. In this case the formula to estimate the magnitude of the step is written down from the condition of stationarity of the functional (23) in α:

$$\alpha = \frac{\sum_{i=1}^{n}\int_0^{t_p}[T^i(1, t, q_0^s) - f_i(t)]\, \vartheta^i(1, t, r^s)\, dt}{\sum_{i=1}^{n}\int_0^{t_p}[\vartheta^i(1, t, r^s)]^2 dt}. \tag{24}$$

The equations of the boundary-value problem in the computation of $\vartheta^i(z_i, t)$, $i = \overline{1, n}$, for the IHCP formulation under consideration are written analogously to the equation in [2].

In conformity with the above, a computational algorithm was developed.

To estimate the influence of the number of temperature measurement points on the convergence of the algorithm and on the accuracy of determining the boundary condition, computations of a number of methodological examples were performed. The exact data on the temperatures at the plate inner points were here obtained by solving the direct heat-conduction

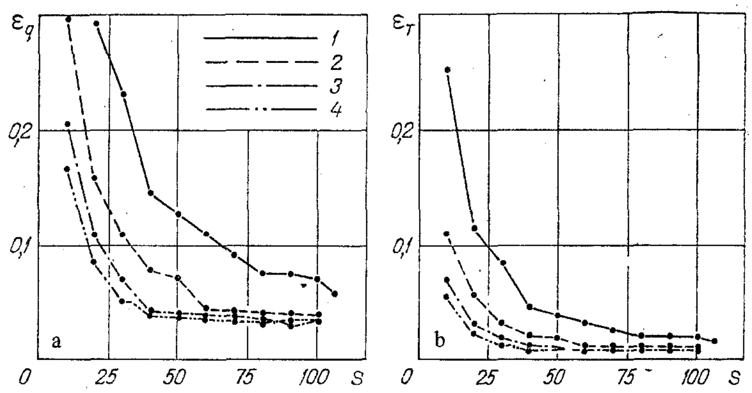

Fig. 1. Change in the error of determining the thermal flux density (a) and the surface temperature (b) as the number of the iteration changes for the case of perturbed initial data: 1) one temperature sensor; 2) two; 3) three; 4) four.

TABLE 1. Dependence of the Errors in Determining the Boundary Functions on the Number of Sensors

No. of sensors	Sensor location coordinates $\overline{X}_i = X_i/(at_p)^{1/2}$						ε_q	ε_T
	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$		
1	0,058	—	—	—	—	—	0,140	0,045
2	0,058	0,116	—	—	—	—	0,056	0,011
3	0,058	0,116	0,232	—	—	—	0,039	0,009
4	0,058	0,116	0,232	0,348	—	—	0,025	0,006
5	0,058	0,116	0,232	0,348	0,464	—	0,015	0,003
6	0,058	0,116	0,232	0,348	0,464	0,58	0,016	0,004

problem with given boundary conditions. Then the time dependences of the temperature at a number of inner points were used as input information for the solution of the IHCP on finding the now unknown flux density $q_0(t)$ (and the surface temperature $T^1(0, t)$ simultaneously). Such an approach permits execution of a direct comparison between the known thermal flux density and its values obtained by solving the inverse problem.

The errors in determining the boundary functions, the thermal flux density q and the surface temperature T, were here estimated from the formula

$$\varepsilon_y = \left[\int_0^{t_p} [y(t) - \tilde{y}(t)]^2 dt \Big/ \int_0^{t_p} y^2(t)\, dt\right]^{1/2}, \tag{25}$$

where y(t) is the exact dependence and $\tilde{y}(t)$ is the dependence obtained from the solution of the inverse problem.

Data on the change in the errors mentioned as a function of the number of temperature sensors whose measurement results were taken into account in solving the IHCP are represented in Table 1. The number of iterations was here limited to a number equal to the number of nodes for the discrete representation of the desired function $q_0(t)$ ($l = 50$). These data permit indirect estimation of the change in the rate of convergence of the algorithm as the number of sensors changes.

Moreover, the influence of the number of temperature sensors was estimated for the limited accuracy of the temperature measurements. Random errors were modeled by using random sensors. Uniform and normal distribution laws were hence simulated.

As an illustration, the change in the errors ε_q and ε_T during an iterative search for the solution of the inverse problem is shown in the figure for a normal distribution law for a different number of temperature sensors. The maximal error in the temperature measurement is here 5% of $f_i^{max}(t)$.

The results of the computations performed showed that an increase in the number of temperature sensors results not only in a rise in the rate of convergence of the algorithm but also in a rise in the stability to perturbations in the initial data.

In conclusion, it must be noted that the scheme considered above for $n \geq 2$ can be used to determine the boundary conditions on both boundaries. To do this, it is just necessary to write down the formula for J'_{q_n}.

NOTATION

n, number of temperature measurement points; τ, t, time; τ_p, t_p, length of the time interval; x, a space coordinate; $X_i(t)$, $i = \overline{1, n}$, coordinates of the temperature measurement points; T(x, t), temperature; C(T), bulk specific heat of the material; λ(T), coefficient of material heat conduction; T(x, 0), initial temperature distribution; q, heat flux density; $f_i(t)$, $i = \overline{1, n}$, temperature measurement; $\Psi^i(z_i, t)$, $i = \overline{1, n}$, conjugate variable; $\vartheta^i(z_i, t)$, $i = \overline{1, n}$, temperature variation; α, β, p, parameters of the conjugate gradient method; s, number of iteration; l, number of points in a discrete representation of the time function; ε, an error estimate.

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